

Existence, uniqueness, approximation of solutions and \mathbb{E}_α -Ulam stability results for a class of nonlinear fractional differential equations involving ψ -Caputo derivative with initial conditions

CHOUKRI DERBAZI, ZIDANE BAITICHE,
MOUFFAK BENCHOHRA, GASTON N'GUÉRÉKATA*

ABSTRACT. The main purpose of this paper is to study the existence, uniqueness, \mathbb{E}_α -Ulam stability results, and other properties of solutions for certain classes of nonlinear fractional differential equations involving the ψ -Caputo derivative with initial conditions. Modern tools of functional analysis are applied to obtain the main results. More precisely using Weisinger's fixed point theorem and Schaefer's fixed point theorem the existence and uniqueness results of solutions are proven in the bounded domain. While the well known Banach fixed point theorem coupled with Bielecki type norm are used with the end goal to establish sufficient conditions for existence and uniqueness results on unbounded domains. Meanwhile, the monotone iterative technique combined with the method of upper and lower solutions is used to prove the existence and uniqueness of extremal solutions. Furthermore, by means of new generalizations of Gronwall's inequality, different kinds of \mathbb{E}_α -Ulam stability of the proposed problem are studied. Finally, as applications of the theoretical results, some examples are given to illustrate the feasibility and correctness of the main results.

1. INTRODUCTION

Fractional differential equations have appeared in many branches of physics, economics, and technical sciences [32, 44, 45, 46]. More details on the fundamental concepts of fractional calculus can be found in the monographs [33, 40, 42]. For some recent contributions on fractional differential equations, we refer the reader to the books of Abbas et al. [1, 2, 3], and Zhou [56, 57]. Recently, in [10] Almeida defined a new fractional derivative

2020 *Mathematics Subject Classification.* Primary: 34A08; Secondary: 26A33.

Key words and phrases. ψ -Caputo fractional derivative, fixed point, existence, uniqueness, extremal solutions, \mathbb{E}_α -Ulam stability, Bielecki norm, monotone iterative technique, upper and lower solutions.

Full paper. Received 22 August 2020, accepted 14 September 2020, available online 16 March 2021.

*Corresponding Author.

with a kernel of a function called the ψ -fractional derivative. The reader interested in the subject of initial and boundary value problems involving ψ -fractional derivative is referred to as [8, 11, 12, 13, 14], and the references given therein.

In recent years, there are several contributions focusing on fractional differential equations, mainly on the existence, uniqueness, and stability of solutions. For more details, the readers are referred to the previous studies [4, 5, 6, 7, 9, 15, 17, 18, 19, 20, 21, 22, 23, 24, 35, 36, 41, 43, 49, 50, 51, 52, 53, 54, 55] and the references therein. The main techniques used in these studies are fixed-point techniques, Leray-Schauder theory, coincidence theory, or monotone iterative technique combined with the method of upper and lower solutions.

In 2008, Benchohra et al. [18], considered the following initial value problem of the form

$$\begin{cases} {}^c\mathcal{D}_{0+}^\alpha u(t) = f(t, u(t)), & 1 < \alpha \leq 2, \quad t \in [0, T], \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

where ${}^c\mathcal{D}_{0+}^\alpha$ is the Caputo fractional derivative $f : [0, T] \times E \rightarrow E$ is a given function, and E is a Banach space. The authors obtained the existence of solutions by means of Mönch's fixed point theorem combined with the technique of measure of noncompactness.

In [38], Matar discussed the existence and uniqueness of the positive solution of the following nonlinear fractional differential equation

$$\begin{cases} {}^c\mathcal{D}_{0+}^\alpha u(t) = f(t, u(t)), & 1 < \alpha \leq 2, \quad t \in [0, 1], \\ u(0) = 0, \quad u'(0) = \theta > 0, \end{cases}$$

where ${}^c\mathcal{D}_{0+}^\alpha$ is the Caputo fractional derivative, and $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a given continuous function. By employing the method of the upper and lower solutions, Schauder and Banach fixed point theorems, the author obtained positivity results.

In [27], Ge and Kou investigated the asymptotic stability of the zero solution of the following nonlinear fractional differential equation

$$\begin{cases} {}^c\mathcal{D}_{0+}^\alpha u(t) = f(t, u(t)), & 1 < \alpha \leq 2, \quad t \geq 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

where ${}^c\mathcal{D}_{0+}^\alpha$ is the Caputo fractional derivative, and $f : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function. They obtained the stability results by employing the Krasnoselskii's fixed point theorem in a weighted Banach space. Furthermore, Hallaci et al [31] studied the uniqueness of solutions to the above problem by means of Banach's contraction principle.

In [16] Ardjouni et al. gave sufficient conditions to guarantee the asymptotic stability of the zero solution to a kind of higher-order nonlinear

fractional differential equations. By using Krasnoselskii's fixed point theorem in a weighted Banach space and they generalized the work of Ge and Kou [27]. More precisely, they studied the following higher-order nonlinear fractional differential equations of the form

$$\begin{cases} {}^c\mathcal{D}_{0+}^\alpha u(t) = f(t, u(t)), & t \geq 0, \\ u^{(i)}(0) = u_i, & i = 0, \dots, n, n \geq 1 \end{cases}$$

where ${}^c\mathcal{D}_{0+}^\alpha$ is the Caputo fractional derivative of order $n < \alpha \leq n + 1$, $f : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function and $u_i \in \mathbb{R}$.

Very recently, Matar et al [39] investigated the existence, uniqueness, and stability of solution for the semi-linear fractional differential systems of the form

$$\begin{cases} {}^c\mathcal{D}_{t_0+}^\alpha u(t) = Au(t) + f(t, u(t)), & t \geq t_0, \\ u(t_0) = u_0, & u'(t_0) = u_1, \end{cases}$$

where ${}^c\mathcal{D}_{t_0+}^\alpha$ is the Caputo fractional derivative of order $1 < \alpha \leq 2$, $f : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given continuous function, $A \in \mathbb{R}^{n \times n}$ and $u_0, u_1 \in \mathbb{R}^n$. By means of Schaefer's fixed-point theorem and Banach's contraction principle, the authors obtained the existence and uniqueness of solutions, as well as the stability results.

In recent years, Almeida et al. [12] considered the following ψ -Caputo fractional ordinary differential equations with initial conditions:

$$\begin{cases} {}^c\mathcal{D}_{a+}^{\alpha; \psi} u(t) = f(t, u(t)), & t \in J := [a, b], \\ u(a) = u_a, & u_\psi^{[k]}(a) = u_a^{[k]}, \quad k = 1, \dots, n - 1, \end{cases}$$

where ${}^c\mathcal{D}_{a+}^{\alpha; \psi}$ is the ψ -Caputo fractional derivative of order $n - 1 < \alpha \leq n$, $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function and $u_a, u_a^{[k]} \in \mathbb{R}$. By using some standard fixed point theorems, the authors obtained the existence and uniqueness of solutions.

Inspired by the above works, in this paper, we try to investigate some basic qualitative properties (existence, uniqueness, and stability of solutions) as well as the existence and uniqueness of extremal solution for the following initial value problem of the fractional differential equation of the form

$$(1) \quad \begin{cases} {}^c\mathcal{D}_{a+}^{\alpha; \psi} u(t) = f(t, u(t)), & t \in J := [a, b], \\ u(a) = \theta_0, & u_\psi^{[1]}(a) = \theta_1, \end{cases}$$

where ${}^c\mathcal{D}_{a+}^{\alpha; \psi}$ is the ψ -Caputo fractional derivatives such that $1 < \alpha \leq 2$, $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function satisfying some assumptions that will be specified later and $\theta_0, \theta_1 \in \mathbb{R}$. Moreover, we also extend the above problem to give a uniqueness results in the half line via Banach contraction principle coupled with Bielecki type norm.

Compared with the existing literature, this paper presents the following new features.

- (1) We present a class of fractional differential equations involving the ψ -Caputo fractional derivative on a bounded and unbounded domain.
- (2) From the choice of $\psi(t) = t$ and $\psi(t) = \ln t$, we have the problems with their respective solutions, for the Caputo and Caputo-Hadamard fractional derivatives, respectively. In addition to the integer case, by choosing $\psi(t) = t, \alpha = 2$.
- (3) The main tools used in our study are techniques of nonlinear analysis such as fixed point theorems (Banach's contraction principle, Weissinger's fixed point theorem, Schaefer's fixed point theorem), and the monotone iterative technique combined with the method of upper and lower solutions. Moreover, by means of new generalizations of Gronwall's inequality, we discuss different kinds of \mathbb{E}_α -Ulam stability.
- (4) New comparison results based on new Bielecki-type norm introduced by Sousa et al.[48].
- (5) The conditions imposed on the IVP (1) are weak.
- (6) The results obtained in this paper are generalizations of some results obtained in [31, 38].

2. PRELIMINARIES AND BACKGROUND MATERIALS

In this section, we present some basic notations, definitions, and preliminary results, which will be used throughout this paper.

Before introducing the basic facts on fractional operators, we recall three types of functions that are important in fractional calculus: the Gamma, Beta, and Mittag-Leffler functions.

Definition 1 ([33]). The Gamma function, or second order Euler integral, denoted $\Gamma(\cdot)$ is defined as:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt, \quad \alpha > 0.$$

Definition 2 ([33]). The Beta function, or the first order Euler function, can be defined as

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt, \quad \alpha, \beta > 0.$$

We use the following formula which expresses the beta function in terms of the gamma function:

$$(2) \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \alpha, \beta > 0.$$

The next function is a direct generalization of the exponential series.

Definition 3 ([28]). The one-parameter Mittag-Leffler function $\mathbb{E}_\alpha(\cdot)$, is defined as:

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (z \in \mathbb{R}, \alpha > 0).$$

For $\alpha = 1$, this function coincides with the series expansion of e^z , i.e.,

$$\mathbb{E}_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Definition 4 ([28]). The Two-parameter Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}(\cdot)$, is defined as:

$$(3) \quad \mathbb{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0 \text{ and } z \in \mathbb{R}.$$

Now, we give some results and properties from the theory of fractional calculus. We begin by defining ψ -Riemann-Liouville fractional integrals and derivatives. In what follows,

Definition 5 ([10, 33]). For $\alpha > 0$, the left-sided ψ -Riemann-Liouville fractional integral of order α for an integrable function $u: J \rightarrow \mathbb{R}$ with respect to another function $\psi: J \rightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in J$ is defined as follows

$$(4) \quad \mathcal{I}_{a^+}^{\alpha;\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s) ds,$$

where Γ is the gamma function.

Definition 6 ([10]). Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(J, \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The left-sided ψ -Riemann-Liouville fractional derivative of a function u of order α is defined by

$$\begin{aligned} \mathcal{D}_{a^+}^{\alpha;\psi} u(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \mathcal{I}_{a^+}^{n-\alpha;\psi} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} u(s) ds, \end{aligned}$$

where $n = [\alpha] + 1$.

Definition 7 ([10]). Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(J, \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The left-sided ψ -Caputo fractional derivative of u of order α is defined by

$${}^c \mathcal{D}_{a^+}^{\alpha;\psi} u(t) = \mathcal{I}_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n u(t),$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$.

To simplify notation, we will use the abbreviated symbol

$$(5) \quad u_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n u(t).$$

From the definition, it is clear that

$$(6) \quad {}^c\mathcal{D}_{a^+}^{\alpha;\psi} u(t) = \begin{cases} \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} u_\psi^{[n]}(s) ds & , \text{ if } \alpha \notin \mathbb{N}, \\ u_\psi^{[n]}(t) & , \text{ if } \alpha \in \mathbb{N}. \end{cases}$$

This generalization (6) yields the Caputo fractional derivative operator when $\psi(t) = t$. Moreover, for $\psi(t) = \ln t$, it gives the Caputo–Hadamard fractional derivative.

We note that if $u \in C^n(\mathbb{J}, \mathbb{R})$ the ψ -Caputo fractional derivative of order α of u is determined as

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi} u(t) = \mathcal{D}_{a^+}^{\alpha;\psi} \left[u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(a)}{k!} (\psi(t) - \psi(a))^k \right].$$

(see, for instance, [10, Theorem 3]).

Lemma 1 ([12]). *Let $\alpha, \beta > 0$, and $u \in L^1(\mathbb{J}, \mathbb{R})$. Then*

$$\mathcal{I}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\beta;\psi} u(t) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi} u(t), \text{ a.e. } t \in \mathbb{J}.$$

In particular,

$$\text{If } u \in C(\mathbb{J}, \mathbb{R}). \text{ Then } \mathcal{I}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\beta;\psi} u(t) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi} u(t), t \in \mathbb{J}.$$

Next, we recall the property describing the composition rules for fractional ψ -integrals and ψ -derivatives.

Lemma 2 ([12]). *Let $\alpha > 0$, The following holds:*

If $u \in C(\mathbb{J}, \mathbb{R})$ then

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\alpha;\psi} u(t) = u(t), t \in \mathbb{J}.$$

If $u \in C^n(\mathbb{J}, \mathbb{R})$, $n - 1 < \alpha < n$. Then

$$\mathcal{I}_{a^+}^{\alpha;\psi} {}^c\mathcal{D}_{a^+}^{\alpha;\psi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(a)}{k!} [\psi(t) - \psi(a)]^k,$$

for all $t \in \mathbb{J}$.

Lemma 3 ([12, 33]). *Let $t > a$, $\alpha \geq 0$, and $\beta > 0$. Then*

- (1) $\mathcal{I}_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(a))^{\beta+\alpha-1}$,
- (2) ${}^c\mathcal{D}_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t) - \psi(a))^{\beta-\alpha-1}$,
- (3) ${}^c\mathcal{D}_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^k = 0$, for all $k \in \{0, \dots, n-1\}$, $n \in \mathbb{N}$.

In the sequel, we will make use of the following generalizations of Gronwall's lemmas

Theorem 1 ([47]). *Let x, y be two integrable functions and z continuous, with domain $[a, b]$. Let $\psi \in C^1(\mathbf{J}, \mathbb{R}_+)$ an increasing function such that $\psi'(t) \neq 0, \forall t \in \mathbf{J}$. Assume that*

- (1) *x and y are nonnegative*
- (2) *z is nonnegative and nondecreasing. If*

$$x(t) \leq y(t) + z(t) \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} x(s) ds, \quad t \in \mathbf{J}.$$

Then

$$x(t) \leq y(t) + \int_a^t \sum_{n=0}^{\infty} \frac{(z(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \psi'(s)(\psi(t) - \psi(s))^{n\alpha-1} y(s) ds, \quad t \in \mathbf{J}.$$

Corollary 1 ([47]). *Under the hypotheses of Theorem 1, let y be a nondecreasing function on $[a, b]$. Then, we have*

$$(7) \quad x(t) \leq y(t) \mathbb{E}_\alpha(\Gamma(\alpha)z(t)(\psi(t) - \psi(a))^\alpha), \quad t \in \mathbf{J}.$$

where $\mathbb{E}_\alpha(\cdot)$ is a Mittag-Leffler function with one parameter.

Lemma 4 ([11]). *Let $\alpha, \beta > 0$. Then for all $t \in \mathbf{J}$ we have*

$$\mathcal{I}_{a^+}^{\alpha; \psi} (\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)) = \frac{1}{\beta} (\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha) - 1).$$

Now, we recall some fixed point theorems that will be used later

Theorem 2 (Weissinger's fixed point theorem [26]). *Assume (E, d) to be a non empty complete metric space and let $\beta_j \geq 0$ for every $j \in \mathbb{N}$ such that the series $\sum_{j=0}^{\infty} \beta_j$ converges. Furthermore, let the mapping $\mathbb{T} : E \rightarrow E$ satisfy the inequality*

$$d(\mathbb{T}^j u, \mathbb{T}^j v) \leq \beta_j d(u, v),$$

for every $j \in \mathbb{N}$ and every $u, v \in E$. Then, \mathbb{T} has a unique fixed point u^ . Moreover, for any $v_0 \in E$, the sequence $\{\mathbb{T}^j v_0\}_{j=1}^{\infty}$ converges to this fixed point u^* .*

Theorem 3 (Schaefer's fixed point theorem [29]). *Let X be a Banach space, and $\mathcal{T} : X \rightarrow X$ completely continuous operator. If the set*

$$\Theta = \{u \in X : u = \lambda \mathcal{T}u, \text{ for some } \lambda \in (0, 1)\},$$

is bounded, then \mathcal{T} has fixed points.

To conclude this section, we show the following lemma which has an important role in this paper.

Lemma 5 ([34]). *Suppose that \mathcal{V} is a linear bounded operator defined on a Banach space X , and assume that $\|\mathcal{V}\| < 1$. Then $(I - \mathcal{V})^{-1}$ is linear and bounded. Also*

$$(I - \mathcal{V})^{-1} = \sum_{n=0}^{\infty} \mathcal{V}^n,$$

the convergence of the series being in the operator norm , and

$$\|(I - \mathcal{V})^{-1}\| \leq \frac{1}{1 - \|\mathcal{V}\|}.$$

For the existence of solutions for the problem (1) we need the following lemma:

Lemma 6. *For a given $h \in C(\mathbb{J}, \mathbb{R})$, the unique solution of the linear fractional initial value problem*

$$(8) \quad \begin{cases} {}^c\mathcal{D}_{a^+}^{\alpha;\psi} u(t) = h(t), & t \in \mathbb{J} := [a, b], \\ u(a) = \theta_0, & u_\psi^{[1]}(a) = \theta_1, \end{cases}$$

is given by

$$(9) \quad \begin{aligned} u(t) &= \theta_0 + \theta_1(\psi(t) - \psi(a)) + \mathcal{I}_{a^+}^{\alpha;\psi} h(t) \\ &= \theta_0 + \theta_1(\psi(t) - \psi(a)) + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} h(s) ds. \end{aligned}$$

Proof. Firstly, we apply the ψ -Riemann-Liouville fractional integral of order α to both sides of (8), and then use Lemma 2 to obtain

$$(10) \quad u(t) = \mathcal{I}_{a^+}^{\alpha;\psi} h(t) + k_0 + k_1(\psi(t) - \psi(a)), \quad t \in \mathbb{J} := [a, b],$$

where $k_0, k_1 \in \mathbb{R}$. Since $u(a) = \theta_0$ and $u_\psi^{[1]}(a) = \theta_1$, we deduce that

$$k_0 = \theta_0, \quad k_1 = \theta_1.$$

Substituting the above values of k_0 and k_1 into (10), we get the integral equation (9). The converse follows by direct computation which completes the proof. \square

We introduce the following conditions:

(H1) The function $f : \mathbb{J} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(H2) There exists $L > 0$ such that

$$|f(t, u) - f(t, v)| \leq L|u - v|, \quad t \in \mathbb{J}, u, v \in \mathbb{R}.$$

Now, we are ready to present our main results.

3. EXISTENCE RESULT VIA WEISSINGER'S FIXED POINT THEOREM

The first existence result is based on Weissinger's fixed point theorem.

Theorem 4. *Assume that (H1) and (H2) holds. Then there exists a unique solution of problem (1) on \mathbb{J} .*

Proof. In view of Lemma 6 we define the operator $\mathcal{T} : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ by

$$(11) \quad \begin{aligned} \mathcal{T}u(t) &= \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds \\ &\quad + \theta_0 + \theta_1(\psi(t) - \psi(a)). \end{aligned}$$

It is obvious that \mathcal{T} is well defined due to (H1). Also, from (11) the fixed points of operator \mathcal{T} are solutions of IVP (1).

In order to prove the uniqueness of the solution for the IVP (1), we prove that the operator \mathcal{T} has a unique fixed point. For this, we shall first prove that, for all $n \in \{0, 1, 2, \dots\}$, $t \in J$ and $u, v \in C(J, \mathbb{R})$, the following inequality holds:

$$(12) \quad \|\mathcal{T}^n u - \mathcal{T}^n v\|_\infty \leq \frac{(L(\psi(b) - \psi(a))^\alpha)^n}{\Gamma(n\alpha + 1)} \|u - v\|_\infty.$$

For $n = 0$, this statement is trivially true. We provide the proof of above inequality using mathematical induction. We assume that (12) is true for $n = k$ and prove it for $n = k + 1$. Given $u, v \in C(J, \mathbb{R})$ and $t \in J$, using (H2), we have

$$\begin{aligned} |\mathcal{T}^{k+1}u(t) - \mathcal{T}^{k+1}v(t)| &= |\mathcal{T}(\mathcal{T}^k u(t)) - \mathcal{T}(\mathcal{T}^k v(t))| \\ &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s, \mathcal{T}^k u(s)) - f(s, \mathcal{T}^k v(s))| ds \\ &\leq \frac{L^{k+1} \|u - v\|_\infty}{\Gamma(k\alpha + 1)} \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (\psi(s) - \psi(a))^{k\alpha} ds. \end{aligned}$$

Also note that

$$\begin{aligned} \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (\psi(s) - \psi(a))^{k\alpha} ds &= \frac{(\psi(t) - \psi(a))^{(k+1)\alpha}}{\Gamma(\alpha)} \\ &\times \int_0^1 (1-y)^{\alpha-1} y^{k\alpha} dy \\ &= \frac{(\psi(t) - \psi(a))^{(k+1)\alpha}}{\Gamma(\alpha)} B(\alpha, k\alpha + 1) \\ &= \frac{\Gamma(k\alpha + 1) (\psi(t) - \psi(a))^{(k+1)\alpha}}{\Gamma((k+1)\alpha + 1)}. \end{aligned}$$

where we have used the variable substitution $y = \frac{\psi(s) - \psi(a)}{\psi(t) - \psi(a)}$, and the relationship between the the Beta function and the Gamma function given in (2).

Using the above arguments, we get

$$\|\mathcal{T}^{k+1}u - \mathcal{T}^{k+1}v\|_\infty \leq \frac{(L(\psi(b) - \psi(a))^\alpha)^{(k+1)}}{\Gamma((k+1)\alpha + 1)} \|u - v\|_\infty.$$

Thus, by the principle of mathematical induction on n , the statement (12) is true for each $n \in \mathbb{N}$. We have now shown that the operator \mathcal{T} satisfies the assumptions of Weissinger's fixed point theorem with

$$\beta_n = \frac{(L(\psi(b) - \psi(a))^\alpha)^n}{\Gamma(n\alpha + 1)}.$$

By Definition 3, we have

$$\sum_{n=0}^{\infty} \beta_n = \sum_{n=0}^{\infty} \frac{(L(\psi(b) - \psi(a))^\alpha)^n}{\Gamma(n\alpha + 1)} = \mathbb{E}_\alpha(L(\psi(b) - \psi(a))^\alpha).$$

Hence, by Weissinger's fixed point theorem, \mathcal{T} has a unique fixed point. That is (1) has a unique solution. This completes the proof. \square

An immediate consequence of Theorem 4 is the following result which is useful in the forthcoming analysis.

Corollary 2. *Let $1 < \alpha \leq 2$ and $p, q \in C(J, \mathbb{R})$. Then the following linear fractional initial value problem*

$$(13) \quad \begin{cases} {}^c \mathcal{D}_{a^+}^{\alpha; \psi} u(t) - p(t)u(t) = q(t), & t \in J := [a, b], \\ u(a) = \theta_0, & u_\psi^{[1]}(a) = \theta_1, \end{cases}$$

has a unique solution with the following integral form

$$(14) \quad \begin{aligned} u(t) = & \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \times (q(s) + p(s)u(s)) ds \\ & + \theta_0 + \theta_1(\psi(t) - \psi(a)). \end{aligned}$$

4. EXISTENCE RESULT VIA SCHAEFER'S FIXED POINT THEOREM

The next existence theorem is based on Schaefer's fixed point theorem.

Theorem 5. *Assume that the assumptions (H1) and (H2) are satisfied. Then the IVP (1) has at least one solution defined on J .*

Proof. Let \mathcal{T} be the operator defined in (11). We shall show that the operator \mathcal{T} is completely continuous. We split the proof into several steps.

Step 1: The operator \mathcal{T} is continuous. Suppose that $\{u_n\}$ is a sequence such that $u_n \rightarrow u$ in $C(J, \mathbb{R})$ as $n \rightarrow \infty$. Taking (H2) into consideration we get

$$\begin{aligned} |\mathcal{T}u_n(t) - \mathcal{T}u(t)| & \leq \mathcal{I}_{a^+}^{\alpha; \psi} |f(t, u_n(t)) - f(t, u(t))| \\ & \leq \frac{L(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \|u_n - u\|_\infty. \end{aligned}$$

Hence

$$(15) \quad \|\mathcal{T}u_n - \mathcal{T}u\|_\infty \leq \frac{L(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \|u_n - u\|_\infty.$$

Since $u_n \rightarrow u$ as $n \rightarrow +\infty$, then equation (15) implies

$$\|\mathcal{T}u_n - \mathcal{T}u\|_\infty \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

which implies the continuity of the operator \mathcal{T} .

Step 2: \mathcal{T} maps bounded sets into bounded sets in $C(J, \mathbb{R})$. Indeed, it is enough to show that for any $r > 0$, there exists a positive constant M such that for each $u \in B_r = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \leq r\}$, we have $\|\mathcal{T}u\|_\infty \leq M$. By the assumption (H2), we have

$$\begin{aligned} |\mathcal{T}u(t)| &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) ds \\ &\quad + |\theta_0| + |\theta_1|(\psi(t) - \psi(a)) \\ &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (L|u(s)| + |f(s, 0)|) ds \\ &\quad + |\theta_0| + |\theta_1|(\psi(t) - \psi(a)) \\ &\leq \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} (Lr + f^*) + |\theta_0| + |\theta_1|(\psi(b) - \psi(a)) := M, \end{aligned}$$

where

$$f^* = \sup_{t \in J} |f(t, 0)|.$$

Thus

$$\|\mathcal{T}u\|_\infty \leq M.$$

This proves that \mathcal{T} maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

Step 3: \mathcal{T} maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$. For any $a < t_1 < t_2 < b$ and $u \in B_r$, we get

$$\begin{aligned} &|\mathcal{T}(u)(t_2) - \mathcal{T}(u)(t_1)| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \psi'(s) [(\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}] |f(s, u(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} |f(s, u(s))| ds + |\theta_1| [(\psi(t_2) - \psi(t_1))] \\ &\leq \frac{Lr + f^*}{\Gamma(\alpha + 1)} [(\psi(t_2) - \psi(a))^\alpha - (\psi(t_1) - \psi(a))^\alpha] + |\theta_1| [(\psi(t_2) - \psi(t_1))]. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{T}(u)(t_2) - \mathcal{T}(u)(t_1)\| &\leq \frac{Lr + f^*}{\Gamma(\alpha + 1)} [(\psi(t_2) - \psi(a))^\alpha - (\psi(t_1) - \psi(a))^\alpha] \\ &\quad + |\theta_1| [(\psi(t_2) - \psi(t_1))]. \end{aligned}$$

As $t_2 \rightarrow t_1$, the right-hand side of the above inequality tends to zero independently of $u \in B_r$. As a consequence of Steps 1 to 3 together with the Ascoli–Arzelá theorem, we can conclude that $\mathcal{T} : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ is

completely continuous.

Step 4: A priori bounds.

Now it remains to show that the set

$$\Theta = \{u \in C(J, \mathbb{R}) : u = \lambda \mathcal{T}u, \text{ for some } \lambda \in (0, 1)\},$$

is bounded. Let $u \in C(J, \mathbb{R})$, then $u = \lambda \mathcal{T}u$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$\begin{aligned} |u(t)| &= \lambda |\mathcal{T}u(t)| \leq |\mathcal{T}u(t)| \leq |\theta_0| + |\theta_1|(\psi(t) - \psi(a)) \\ &\quad + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s))| ds. \end{aligned}$$

It follows from assumption (H2), that for every $t \in J$

$$\begin{aligned} |u(t)| &\leq |\theta_0| + |\theta_1|(\psi(t) - \psi(a)) + \frac{f^*(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \\ &\quad + L \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds \\ &= y(t) + L \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds \end{aligned}$$

where $y(t)$ is given by

$$y(t) = |\theta_0| + |\theta_1|(\psi(t) - \psi(a)) + \frac{f^*(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)}.$$

Since $y(t)$ is nondecreasing function on J . Applying the Gronwall inequality Eq. (7), we conclude that

$$|u(t)| \leq y(t) \mathbb{E}_\alpha (L(\psi(t) - \psi(a))^\alpha), \quad t \in J.$$

Hence

$$\|u\| \leq y(b) \mathbb{E}_\alpha (L(\psi(b) - \psi(a))^\alpha).$$

This shows that the set Θ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that \mathcal{T} has a fixed point which is a solution of problem (1). \square

5. EXISTENCE RESULT VIA MONOTONE ITERATIVE TECHNIQUE.

In this section, we prove the existence and uniqueness of solution for (11) by monotone iterative technique combined with the method of upper and lower solutions.

Definition 8. A function $v_0 \in C(J, \mathbb{R})$ is called a lower solution of (11), if it satisfies

$$\begin{aligned} (16) \quad v_0(t) &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, v_0(s)) ds \\ &\quad + \theta_0 + \theta_1(\psi(t) - \psi(a)). \end{aligned}$$

It is said to be an upper solution if the inequality is reversed.

Now, we prove the following comparison result, which plays an important role in our further discussion.

Lemma 7 (Comparison result). *Let $\alpha \in (1, 2]$ be fixed and $p \in C(J, \mathbb{R}^+)$. If $\gamma \in C(J, \mathbb{R})$ satisfies the following inequalities*

$$(17) \quad \gamma(t) \leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} p(s)\gamma(s) ds, \quad t \in J,$$

then $\gamma(t) \leq 0$ for all $t \in J$.

Proof. Consider the Banach space $C(J, \mathbb{R})$ equipped with a Bielecki norm type $\|\cdot\|_{\mathfrak{B}}$ defined as below

$$\|v\|_{\mathfrak{B}} := \sup_{t \in J} \frac{|v(t)|}{\mathbb{E}_{\alpha}(\beta(\psi(t) - \psi(a))^{\alpha})},$$

where $\mathbb{E}_{\alpha}(\cdot)$ is the Mittag-Leffler function which is given in Definition 3. (for more properties on Bielecki type norm see [25, 48]).

Define an operator $\mathcal{V} : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ by

$$(18) \quad \mathcal{V}\gamma(t) = \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} p(s)\gamma(s) ds.$$

Note that $\mathcal{V}\gamma(t) \leq 0$ for all $\gamma \in C(J, \mathbb{R})$ if $\gamma(t) \leq 0$, $t \in J$. Now

$$\begin{aligned} |\mathcal{V}\gamma(t)| &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} p(s)|\gamma(s)| ds \\ &\leq p^* \|\gamma\|_{\mathfrak{B}} \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathbb{E}_{\alpha}(\beta(\psi(s) - \psi(a))^{\alpha}) ds, \end{aligned}$$

where $p^* = \sup_{t \in J} p(t)$. Also note that

$$\mathcal{I}_{a^+}^{\alpha; \psi} (\mathbb{E}_{\alpha}(\beta(\psi(t) - \psi(a))^{\alpha})) = \frac{1}{\beta} (\mathbb{E}_{\alpha}(\beta(\psi(t) - \psi(a))^{\alpha}) - 1).$$

Using the above arguments, we have

$$\|\mathcal{V}\gamma\|_{\mathfrak{B}} \leq \frac{p^* \|\gamma\|_{\mathfrak{B}}}{\beta} \left(1 - \frac{1}{\mathbb{E}_{\alpha}(\beta(\psi(t) - \psi(a))^{\alpha})} \right).$$

Since $\mathbb{E}_{\alpha}(\cdot)$ is a monotone increasing function on J , we have

$$\|\mathcal{V}\gamma\|_{\mathfrak{B}} \leq \frac{p^* \|\gamma\|_{\mathfrak{B}}}{\beta}.$$

which implies that

$$\|\mathcal{V}\| \leq \frac{p^*}{\beta}.$$

Since we can choose $\beta > 0$ sufficiently large such that

$$\frac{p^*}{\beta} < 1,$$

it follows that

$$\|\mathcal{V}\| < 1.$$

Hence, by Lemma 5, we conclude that $(I - \mathcal{V})^{-1}$ is a bounded linear operator satisfying

$$(I - \mathcal{V})^{-1} = \sum_{n=0}^{\infty} \mathcal{V}^n.$$

Now, if $u(t) \leq 0, t \in J$. Then

$$(I - \mathcal{V})^{-1}u(t) = \sum_{n=0}^{\infty} \mathcal{V}^n u(t) \leq 0.$$

Moreover, Eq (17) can be written as

$$(I - \mathcal{V})\gamma(t) \leq 0.$$

Applying $(I - \mathcal{V})^{-1}$ on both sides of the above inequality, one can get

$$\gamma(t) \leq 0. \quad \square$$

The following functional interval plays a fundamental role in our discussion.

$$[v_0, w_0] = \{u \in C(J, \mathbb{R}) : v_0(t) \leq u(t) \leq w_0(t), \quad t \in J\}.$$

In this section, we will apply the monotone iterative method to present a result of the existence and uniqueness of the solution of equation (11).

Theorem 6. *Assume that (H1) holds. In addition assume that:*

- (H3) *There exist $v_0, w_0 \in C(J, \mathbb{R})$ such that v_0 and w_0 are lower and upper solutions of (11), respectively, with $v_0(t) \leq w_0(t), t \in J$.*
- (H4) *There exists $p \in C(J, \mathbb{R}^+)$ such that*

$$f(t, v(t)) - f(t, u(t)) \geq p(t)(v(t) - u(t)), \quad t \in J,$$

$$\text{and } v_0(t) \leq u(t) \leq v(t) \leq w_0(t).$$

Then there exist monotone iterative sequences $\{v_n\}, \{w_n\} \subset [v_0, w_0]$ such that $v_n \rightarrow v^, w_n \rightarrow w^*$ as $n \rightarrow \infty$ uniformly in $[v_0, w_0]$, and v^*, w^* are a minimal and a maximal solution the (11) in $[v_0, w_0]$, respectively and the following relation holds*

$$(19) \quad \begin{aligned} v_0(t) \leq v_1(t) \leq \dots \leq v_n(t) \leq \dots \\ \leq w_n(t) \leq \dots \leq w_1(t) \leq w_0(t), \quad t \in J. \end{aligned}$$

Proof. First, for any $v_0(t), w_0(t) \in C(J, \mathbb{R})$, we define two sequences $\{v_n\}, \{w_n\}$ satisfying the following linear integral equation:

$$(20) \quad \begin{aligned} v_{n+1}(t) = & \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (f(s, v_n(s)) + p(s)(v_{n+1}(s) - v_n(s))) ds \\ & + \theta_0 + \theta_1(\psi(t) - \psi(a)) \quad t \in J, \end{aligned}$$

and

$$(21) \quad \begin{aligned} w_{n+1}(t) = & \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (f(s, w_n(s)) + p(s)(w_{n+1}(s) - w_n(s))) ds \\ & + \theta_0 + \theta_1(\psi(t) - \psi(a)), \quad t \in J. \end{aligned}$$

By Corollary 2, we know that (20) and (21) have unique solutions in $C(J, \mathbb{R})$.

For clarity, we will divide the remain of the proof into several steps.

Step 1: We show that the sequences $v_n(t), w_n(t)$ ($n \geq 1$) are lower and upper solutions of (11), respectively and the relation (19) holds.

First, we prove that

$$(22) \quad v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t), \quad t \in J.$$

Set $\gamma(t) = v_0(t) - v_1(t)$. From (20) and Definition 8, we obtain

$$\gamma(t) \leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} p(s)\gamma(s) ds, \quad t \in J.$$

By Lemma 7, $\gamma(t) \leq 0$, for $t \in J$. That is, $v_0(t) \leq v_1(t)$. Similarly, we can show that $w_1(t) \leq w_0(t)$, $t \in J$.

Now, let $\gamma(t) = v_1(t) - w_1(t)$. From (20), (21) and (H4), we get

$$\begin{aligned} \gamma(t) &= \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (f(s, v_0(s)) - f(s, w_0(s)) \\ &\quad + p(s)(w_0(s) - v_0(s))) ds \\ &\quad + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} p(s)(v_1(s) - w_1(s)) ds \\ &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} p(s)\gamma(s) ds. \end{aligned}$$

By Lemma 7, we get $v_1(t) \leq w_1(t)$, $t \in J$.

Secondly, we show that $x_1(t), y_1(t)$ are lower and upper solutions of (11), respectively. Since v_0 and w_0 are lower and upper solutions of (11), by (H4), it follows that

$$\begin{aligned} v_1(t) &= \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (f(s, v_0(s)) + p(s)(v_1(s) - v_0(s))) ds \\ &\quad + \theta_0 + \theta_1(\psi(t) - \psi(a)) \\ &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, v_1(s)) ds + \theta_0 + \theta_1(\psi(t) - \psi(a)). \end{aligned}$$

Therefore, $v_1(t)$ is a lower solution of (11). Similarly, it can be obtained that $w_1(t)$ is an upper solution of (11).

By the above arguments and mathematical induction, we can show that the sequences $v_n(t), w_n(t), (n \geq 1)$ are lower and upper solutions of (11), respectively and the relation (19) holds.

Step 2: The sequences $\{v_n(t)\}, \{w_n(t)\}$ converge uniformly to their limit functions $v^*(t), w^*(t)$, respectively.

Clearly, the uniform boundedness of sequences $\{v_n(t)\}, \{w_n(t)\}$ follows from (19). Moreover, From the proof of Theorem 5 (Step 3), we can conclude that sequences $\{v_n(t)\}, \{w_n(t)\}$ are equicontinuous. Hence by Ascoli–Arzelá theorem $\{v_n(t)\}, \{w_n(t)\}$ have a convergent subsequences. Combining this with the monotonicity of (19) it follows that $\{v_n(t)\}, \{w_n(t)\}$ are uniformly convergent. That is,

$$\lim_{n \rightarrow \infty} v_n(t) = v^*(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} w_n(t) = w^*(t), \quad t \in \mathbb{J}$$

and the limit functions v^*, w^* satisfy (11).

Step 3: We prove that v^* and w^* are extremal solutions of (11) in $[v_0, w_0]$.

Let $u \in [v_0, w_0]$ be any solution of (11). We assume that the following relation holds for some $n \in \mathbb{N}$:

$$(23) \quad v_n(t) \leq u(t) \leq w_n(t), \quad t \in \mathbb{J}.$$

Let $\gamma(t) = v_{n+1}(t) - u(t)$. Then by (H4), we have

$$\begin{aligned} \gamma(t) &= \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (f(s, v_n(s)) - f(s, u(s))) ds \\ &\quad + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} p(s)(v_{n+1}(s) - v_n(s)) ds \\ &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} p(s)\gamma(s) ds. \end{aligned}$$

Invoking Lemma 7, we conclude that $\gamma(t) \leq 0, t \in \mathbb{J}$, which means

$$v_{n+1}(t) \leq u(t), \quad t \in \mathbb{J}.$$

Using the same method, we can show that

$$u(t) \leq w_{n+1}(t), \quad t \in \mathbb{J}.$$

Hence, we have

$$v_{n+1}(t) \leq u(t) \leq w_{n+1}(t), \quad t \in \mathbb{J}.$$

Therefore, (23) holds on \mathbb{J} for all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ on both sides of (23), we get

$$v^* \leq u \leq w^*.$$

Therefore v^*, w^* are the extremal solutions of (11) in $[v_0, w_0]$. Clearly v^* and w^* are the solutions of (1). This completes the proof. \square

Now, we shall prove the uniqueness of the solution of the the problem (11) by monotone iterative technique.

Theorem 7. *Assume that all assumptions of Theorem 6 hold. In addition, we assume that*

(H5) *There exists $\tilde{p} \in C(J, \mathbb{R}^+)$ such that*

$$f(t, v(t)) - f(t, u(t)) \leq \tilde{p}(t)(v(t) - u(t)), \quad t \in J,$$

and $v_0(t) \leq u(t) \leq v(t) \leq w_0(t)$.

Then (11) has a unique solution between v_0 and w_0 , which can be obtained by a monotone iterative procedure starting from v_0 or w_0 .

Proof. From the Theorem 6, we know that $v^*(t)$ and $w^*(t)$ are the extremal solutions of (11) and $v^*(t) \leq w^*(t)$, $t \in J$. It is sufficient to prove $v^*(t) \geq w^*(t)$, $t \in J$. In fact, let $\gamma(t) = w^*(t) - v^*(t)$, $t \in J$, in view of (H5), we have

$$\begin{aligned} \gamma(t) &= \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (f(s, w^*(s)) - f(s, v^*(s))) ds \\ &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \tilde{p}(s)\gamma(s) ds. \end{aligned}$$

From Lemma 7, it follows that $\gamma(t) \leq 0$, $t \in J$. Hence, we obtain

$$v^*(t) \geq w^*(t), \quad t \in J.$$

Therefore, $v^* \equiv w^*$ is the unique solution of (11) in $[v_0, w_0]$. That is the Cauchy problem (1) has a unique solution in $[v_0, w_0]$. This ends the proof of Theorem 7. \square

6. UNIQUENESS RESULT ON AN UNBOUNDED DOMAIN

By using Banach's fixed point theorem, we give in this subsection our last uniqueness theorem concerning the IVP (1) on unbounded domain i.e in the case $J' = [a, +\infty)$. Before proceeding to the main results, we start by the following lemma.

Lemma 8. *Let*

$$\mathbb{X} = \left\{ v \in C(J', \mathbb{R}) : \sup_{t \geq a} \frac{v(t)}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} < +\infty, \beta > 0 \right\}.$$

equipped with a norm,

$$\|v\|_{\mathbb{X}} = \sup_{t \geq a} \frac{|v(t)|}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)}.$$

Then, $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a Banach space.

Proof. We only prove that the space \mathbb{X} is complete. To do this, let $\{v_n\}$ be a Cauchy sequence in the space \mathbb{X} . Then, for any given $\varepsilon > 0$ and any $t \geq a$, there exists a constant $n_\varepsilon > 0$, such that for $n, m \geq n_\varepsilon$, we have,

$$\frac{|v_n(t) - v_m(t)|}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} \leq \|v_n - v_m\| < \varepsilon,$$

so $\{v_n(t)/\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)\}$ is a Cauchy sequence in \mathbb{R} . So, there exists a $\tilde{v}(t)$ such that

$$\lim_{n \rightarrow +\infty} \frac{v_n(t)}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} = \tilde{v}(t), \quad t \geq a.$$

Let $\varepsilon > 0$; there exists $n_\varepsilon \in \mathbb{N}$ such that for $k, l \geq n_\varepsilon$, we have $\|v_k - v_l\| < \varepsilon$; therefore

$$\frac{|v_k(t) - v_l(t)|}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} < \varepsilon, \quad \text{for any } t \geq a,$$

let $l \rightarrow +\infty$, we have

$$(24) \quad \left| \frac{v_k(t)}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} - \tilde{v}(t) \right| < \varepsilon, \quad t \geq a, \quad k \geq n_\varepsilon.$$

So, $\|v_k - v\| \leq \varepsilon$, where $v(t) = \tilde{v}(t)\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)$. On the other hand,

$$\begin{aligned} \sup_{t \geq a} \frac{|v(t)|}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} &\leq \sup_{t \geq a} \left| \tilde{v}(t) - \frac{v_k(t)}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} \right| \\ &\quad + \frac{|v_k(t)|}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} < +\infty. \end{aligned}$$

It remains to be shown that $v(t) \in C[a, +\infty)$, so it is sufficient to prove that $\tilde{v}(t) \in C[a, +\infty)$. From the continuity of $v_k(t)/\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)$ on $[a, +\infty)$ and (24), for any $t' \in [a, +\infty)$, there exists a $\delta > 0$ such that $|t - t'| < \delta$ implies

$$(25) \quad \begin{aligned} |\tilde{v}(t) - \tilde{v}(t')| &\leq \left| \tilde{v}(t) - \frac{v_k(t)}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} \right| \\ &\quad + \left| \frac{v_k(t)}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} - \frac{v_k(t')}{\mathbb{E}_\alpha(\beta(\psi(t') - \psi(a))^\alpha)} \right| \\ &\quad + \left| \frac{v_k(t')}{\mathbb{E}_\alpha(\beta(\psi(t') - \psi(a))^\alpha)} - \tilde{v}(t') \right| \\ &\leq 2 \sup_{t \geq a} \left| \tilde{v}(t) - \frac{v_k(t)}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} \right| + \varepsilon \leq 3\varepsilon. \end{aligned}$$

Thus, $\tilde{v}(t) \in C[a, +\infty)$. □

To simplify the notation, we will use the abbreviated symbol

$$\mathcal{G}_\psi^\alpha(t, s) = \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}.$$

Theorem 8. *Assume that (H1) and (H2) holds for each $t \in J'$. Then there exists a unique solution of problem (1) on $J' := [a, \infty)$.*

Proof. In view of Lemma 6 we define the operator $\mathcal{T}: \mathbb{X} \rightarrow \mathbb{X}$ by

$$\mathcal{T}u(t) = \theta_0 + \theta_1(\psi(t) - \psi(a)) + \int_a^t \mathcal{G}_\psi^\alpha(t, s)f(s, u(s))ds, \quad t \in J'.$$

In order to prove uniqueness of the solution for the IVP (1) on J' , we prove that the operator \mathcal{T} has a unique fixed point. For this, we shall show that \mathcal{T} is a contraction. Given $u, v \in \mathbb{X}$ and $t \in J'$, using (H2), and Lemma 4, we have

$$\begin{aligned} |\mathcal{T}u(t) - \mathcal{T}v(t)| &\leq \int_a^t \mathcal{G}_\psi^\alpha(t, s) \frac{L|u(s) - v(s)|}{\mathbb{E}_\alpha(\beta(\psi(s) - \psi(a))^\alpha)} \mathbb{E}_\alpha(\beta(\psi(s) - \psi(a))^\alpha) ds \\ &\leq \frac{L}{\beta} (\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha) - 1) \|u - v\|_{\mathbb{X}}. \end{aligned}$$

Hence, we have

$$\|\mathcal{T}u - \mathcal{T}v\|_{\mathbb{X}} \leq \frac{L}{\beta} \left(1 - \frac{1}{\mathbb{E}_\alpha(\beta(\psi(t) - \psi(a))^\alpha)} \right) \|u - v\|_{\mathbb{X}}.$$

Note that $\mathbb{E}_\alpha(\cdot)$ is a monotone increasing function on J' , then we get

$$\|\mathcal{T}u - \mathcal{T}v\|_{\mathbb{X}} \leq \frac{L}{\beta} \|u - v\|_{\mathbb{X}}.$$

Since we can choose $\beta > 0$ sufficiently large such that

$$\frac{L}{\beta} < 1,$$

it follows that

$$\|\mathcal{T}u - \mathcal{T}v\|_{\mathbb{X}} \leq \|u - v\|_{\mathbb{X}}.$$

It follows that the mapping \mathcal{T} is a contraction. Hence, by the Banach fixed point theorem, \mathcal{T} has a unique fixed point which is a unique solution of the initial value problem (1) on $J' := [a, \infty)$. This completes the proof. \square

7. \mathbb{E}_α -ULAM-HYERS STABILITY ANALYSIS

In this section, we study \mathbb{E}_α -Ulam-Hyers, generalized \mathbb{E}_α -Ulam-Hyers, \mathbb{E}_α -Ulam-Hyers-Rassias, and generalized \mathbb{E}_α -Ulam-Hyers-Rassias stability of problem (1).

Now, we consider the \mathbb{E}_α -Ulam-Hyers stability for problem (1). Let $\varepsilon > 0, L_f \geq 0$ and $\Phi : J \rightarrow \mathbb{R}^+$, be a continuous function. We consider the following inequalities:

$$(26) \quad |{}^c\mathcal{D}_{a^+}^{\alpha;\psi} v(t) - f(t, v(t))| \leq \varepsilon, \quad t \in J;$$

$$(27) \quad |{}^c\mathcal{D}_{a^+}^{\alpha;\psi} v(t) - f(t, v(t))| \leq \Phi(t), \quad t \in J;$$

$$(28) \quad |{}^c\mathcal{D}_{a^+}^{\alpha;\psi} v(t) - f(t, v(t))| \leq \varepsilon\Phi(t), \quad t \in J.$$

Definition 9 ([51, 52]). Equation (1) is \mathbb{E}_α -Ulam-Hyers stable if there exists a real number $c > 0$ such that, for each $\varepsilon > 0$ and for each solution $v \in C(J, \mathbb{R})$ of inequalities (26), there exists a solution $u \in C(J, \mathbb{R})$ of (1) with

$$|v(t) - u(t)| \leq c\varepsilon\mathbb{E}_\alpha(L_f(\psi(t) - \psi(a))^\alpha), \quad t \in J.$$

Definition 10 ([51, 52]). Equation (1) is generalized \mathbb{E}_α -Ulam-Hyers stable if there exists $\omega : C(\mathbb{R}_+, \mathbb{R}_+)$ with $\omega(0) = 0$ such that, for each $\varepsilon > 0$ and for each solution $v \in C(J, \mathbb{R})$ of inequalities (26), there exists a solution $u \in C(J, \mathbb{R})$ of (1) with

$$|v(t) - u(t)| \leq \omega(\varepsilon)\mathbb{E}_\alpha(L_f(\psi(t) - \psi(a))^\alpha), \quad t \in J.$$

Definition 11 ([51, 52]). Equation (1) is \mathbb{E}_α -Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_\Phi > 0$ such that, for each $\varepsilon > 0$ and for each solution $v \in C(J, \mathbb{R})$ of inequalities (28), there exists a solution $u \in C(J, \mathbb{R})$ of (1) with

$$|v(t) - u(t)| \leq c_\Phi\varepsilon\Phi(t)\mathbb{E}_\alpha(L_f(\psi(t) - \psi(a))^\alpha), \quad t \in J.$$

Definition 12 ([51, 52]). Equation (1) is generalized \mathbb{E}_α -Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_\Phi > 0$ such that, for each solution $v \in C(J, \mathbb{R})$ of inequalities (27), there exists a solution $u \in C(J, \mathbb{R})$ of (1) with

$$|v(t) - u(t)| \leq c_\Phi\Phi(t)\mathbb{E}_\alpha(L_f(\psi(t) - \psi(a))^\alpha), \quad t \in J.$$

Remark 1 ([51, 52]). It is clear that

- (i) Definition 9 \Rightarrow Definition 10,
- (ii) Definition 11 \Rightarrow Definition 12,
- (iii) Definition 11 for $\Phi(\cdot) = 1 \Rightarrow$ Definition 9.

Remark 2 ([51, 52]). A function $v \in C(J, \mathbb{R})$ is a solution of inequality (26) if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depends on solution v) such that

- (i) $|g(t)| \leq \varepsilon, \quad t \in J.$
- (ii) ${}^c\mathcal{D}_{a^+}^{\alpha;\psi} v(t) = f(t, v(t)) + g(t), \quad t \in J.$

Remark 3 ([51, 52]). A function $v \in C(J, \mathbb{R})$ is a solution of inequality (28) if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depends on solution v) such that

- (i) $|g(t)| \leq \varepsilon \Phi(t), \quad t \in J.$
- (ii) ${}^c \mathcal{D}_{a^+}^{\alpha; \psi} v(t) = f(t, v(t)) + g(t), \quad t \in J.$

Now, we discuss the \mathbb{E}_α -Ulam-Hyers stability of solution to the problem (1). The arguments are based on the Gronwall inequality Eq. (7)

Theorem 9. *Assume that (H1) and (H2) hold. Then problem (1) is \mathbb{E}_α -Ulam-Hyers stable on J and consequently generalized \mathbb{E}_α -Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and let $v \in C(J, \mathbb{R})$ be a function which satisfies the inequality (26) and let $u \in C(J, \mathbb{R})$ the unique solution of the following problem

$$\begin{cases} {}^c \mathcal{D}_{a^+}^{\alpha; \psi} u(t) = f(t, u(t)), & t \in J := [a, b], \\ u(a) = \theta_0, \quad u_\psi^{[1]}(a) = \theta_1. \end{cases}$$

By Lemma 6, we have

$$u(t) = \theta_0 + \theta_1(\psi(t) - \psi(a)) + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds.$$

Since we have assumed that v is a solution of (26), hence we have by Remark 2.

$$\begin{cases} {}^c \mathcal{D}_{a^+}^{\alpha; \psi} v(t) = f(t, v(t)) + g(t), & t \in J := [a, b], \\ v(a) = \theta_0, \quad v_\psi^{[1]}(a) = \theta_1. \end{cases}$$

Again by Lemma 6, we have

$$v(t) = \theta_0 + \theta_1(\psi(t) - \psi(a)) + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (f(s, v(s)) + g(s)) ds.$$

On the other hand, we have, for each $t \in J$

$$\begin{aligned} |v(t) - u(t)| &\leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |g(s)| ds \\ &\quad + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s, v(s)) - f(s, u(s))| ds. \end{aligned}$$

Hence using part (i) of Remark 2, and (H2) we can get

$$|v(t) - u(t)| \leq \frac{(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \varepsilon + L \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |v(s) - u(s)| ds.$$

Applying Corollary 1 (the Gronwall inequality Eq. (7)), to above inequality with $x(t) = |v(t) - u(t)|$, $y(t) = \frac{(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \varepsilon$ and $z(t) = \frac{L}{\Gamma(\alpha)}$. Since $y(t)$ is nondecreasing function on J , we conclude that

$$|v(t) - u(t)| \leq y(t) \mathbb{E}_\alpha (L(\psi(t) - \psi(a))^\alpha), \quad t \in J.$$

Which yields that

$$(29) \quad |v(t) - u(t)| \leq \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \mathbb{E}_\alpha (L(\psi(t) - \psi(a))^\alpha) \varepsilon, \quad t \in J.$$

Taking for simplicity

$$c = \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)},$$

then (29) becomes

$$|v(t) - u(t)| \leq c\varepsilon \mathbb{E}_\alpha (L(\psi(t) - \psi(a))^\alpha), \quad t \in J.$$

Thus, the problem (1) is \mathbb{E}_α -Ulam-Hyers stable. Further, if we set $\omega(\varepsilon) = c\varepsilon; \omega(0) = 0$, then the problem (1) is generalized \mathbb{E}_α -Ulam-Hyers stable. This completes the proof. \square

Now we are ready to state our \mathbb{E}_α -Ulam-Hyers-Rassias stability result.

Theorem 10. *Assume that (H1) and (H2) hold and the following hypothesis hold:*

(H6) *There exists an increasing function $\Phi \in C(J, \mathbb{R}_+)$ and there exists $\lambda_\Phi > 0$ such that for any $t \in J$*

$$\mathcal{I}_{a^+}^{\alpha; \psi} \Phi(t) \leq \lambda_\Phi \Phi(t).$$

Then, the problem (1) is \mathbb{E}_α -Ulam-Hyers-Rassias stable and consequently generalized \mathbb{E}_α -Ulam-Hyers-Rassias stable.

Proof. Let $\varepsilon > 0$ and let $v \in C(J, \mathbb{R})$ be a function which satisfies the inequality (28) and let $u \in C(J, \mathbb{R})$ the unique solution of the following problem

$$\begin{cases} {}^c \mathcal{D}_{a^+}^{\alpha; \psi} u(t) = f(t, u(t)), & t \in J := [a, b], \\ u(a) = \theta_0, \quad u_\psi^{[1]}(a) = \theta_1. \end{cases}$$

By Lemma 6, we have

$$u(t) = \theta_0 + \theta_1(\psi(t) - \psi(a)) + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds.$$

Since we have assumed that v is a solution of (28), hence we have by Remark 3.

$$\begin{cases} {}^c \mathcal{D}_{a^+}^{\alpha; \psi} v(t) = f(t, v(t)) + g(t), & t \in J := [a, b], \\ v(a) = \theta_0, \quad v_\psi^{[1]}(a) = \theta_1. \end{cases}$$

Again by Lemma 6, we have

$$v(t) = \theta_0 + \theta_1(\psi(t) - \psi(a)) + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (f(s, v(s)) + g(s)) ds.$$

On the other hand, we have, for each $t \in J$

$$|v(t) - u(t)| \leq \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |g(s)| ds$$

$$+ \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s, v(s)) - f(s, u(s))| ds.$$

Hence using part (i) of Remark 3, (H2) and (H6) we can get

$$\begin{aligned} |v(t) - u(t)| &\leq \varepsilon \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \Phi(s) ds \\ &+ L \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |v(s) - u(s)| ds \\ &\leq \varepsilon \lambda_{\Phi} \Phi(t) + \frac{L}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |v(s) - u(s)| ds. \end{aligned}$$

Applying Corollary 1 (the Gronwall inequality Eq. (7)), to above inequality with $x(t) = |v(t) - u(t)|$, $y(t) = \varepsilon \lambda_{\Phi} \Phi(t)$ and $z(t) = \frac{L}{\Gamma(\alpha)}$. Since $y(t)$ is nondecreasing function on J , we conclude that

$$|v(t) - u(t)| \leq y(t) \mathbb{E}_{\alpha} (L(\psi(t) - \psi(a))^{\alpha}), \quad t \in J.$$

Which yields that

$$(30) \quad |v(t) - u(t)| \leq \varepsilon \lambda_{\Phi} \mathbb{E}_{\alpha} (L(\psi(t) - \psi(a))^{\alpha}) \Phi(t), \quad t \in J.$$

Taking for simplicity

$$c_{\Phi} = \lambda_{\Phi},$$

then (30) becomes

$$|v(t) - u(t)| \leq c_{\Phi} \varepsilon \Phi(t) \mathbb{E}_{\alpha} (L(\psi(t) - \psi(a))^{\alpha}), \quad t \in J.$$

Thus, the problem (1) is \mathbb{E}_{α} -Ulam–Hyers–Rassias stable. Further, if we set $\varepsilon = 1$, then the problem (1) is generalized \mathbb{E}_{α} -Ulam–Hyers–Rassias stable. This completes the proof. \square

8. APPLICATIONS

In this section, we are illustrating our theory results by several examples.

Example 1. Let us consider problem (1) with specific data:

$$(31) \quad \alpha = \frac{3}{2}, \quad \theta_0 = 1, \quad \theta_1 = 2 \quad a = 0, \quad b = 1.$$

In order to illustrate Theorem 4, we take $\psi(t) = \sigma(t)$ where $\sigma(t)$ is the Sigmoid function [37] which can be expressed as in the following form

$$(32) \quad \sigma(t) = \frac{1}{1 + e^{-t}},$$

and a convenience of the Sigmoid function is its derivative

$$\sigma'(t) = \sigma(t)(1 - \sigma(t)).$$

Taking also

$$(33) \quad f(t, u(t)) = \frac{1}{e^t + 9} \left(1 + \frac{|u(t)|}{1 + |u(t)|} \right),$$

in (1). Clearly, the function f is continuous. Moreover, For any $u, v \in \mathbb{R}$ and $t \in [0, 1]$ we have

$$\begin{aligned} |f(t, u) - f(t, v)| &= \frac{1}{e^t + 9} \left(\left| \frac{|u|}{1 + |u|} - \frac{|v|}{1 + |v|} \right| \right) \\ &\leq \frac{1}{e^t + 9} \left(\frac{|u - v|}{(1 + |u|)(1 + |v|)} \right) \\ &\leq \frac{1}{10} |u - v|. \end{aligned}$$

Hence the condition (H2) holds with $L = \frac{1}{10}$. It follows from Theorem 4 that the problem (1) with the data (31), (32) and (33) has a unique solution on $[0, 1]$. Moreover, Theorem 9 implies that the problem (1) is \mathbb{E}_α -Hyers-Ulam stable and generalized \mathbb{E}_α -Hyers-Ulam stable.

Example 2. Consider the following initial value problem:

$$(34) \quad \begin{cases} {}^c \mathcal{D}_{0^+}^{\frac{3}{2}} x(t) = 1 + \frac{\sin u(t)}{e^{-t} + 1} + 2t, & t \in J := [0, 1], \\ u(0) = 1 \quad u'(0) = 2. \end{cases}$$

Note that, this problem is a particular case of IVP (1), where

$$\alpha = \frac{3}{2}, \quad a = 0, \quad b = 1, \quad \theta_0 = 1, \quad \theta_1 = 2, \quad \psi(t) = t,$$

and $f: J \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t, u) = 1 + \frac{\sin u(t)}{e^{-t} + 1} + 2t, \quad \text{for } t \in J, u \in \mathbb{R}.$$

It is clear that the function f is continuous which satisfies

$$|f(t, u) - f(t, v)| \leq \frac{1}{2} |u - v|,$$

Hence condition (H2) holds with $L = \frac{1}{2}$. So by Theorem the problem (34) has at least a solution defined on $[0, 1]$.

Example 3. Consider the following fractional IVP:

$$(35) \quad \begin{cases} {}^c {}_H \mathcal{D}_{1^+}^{\frac{3}{2}} u(t) = f(t, u(t)), & t \in J := [1, e], \\ u(1) = 0.5, \quad u'(1) = 1, \end{cases}$$

In this case we take

$$\alpha = \frac{3}{2}, \quad a = 1, \quad b = e, \quad \theta_0 = 0.5, \quad \theta_1 = 1, \quad \psi(t) = \ln t,$$

and $f: [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t, u) = \frac{1}{(1 + t)^2} (1 + \arctan(|u|)), \quad \text{for } t \in [1, e], u \in \mathbb{R}.$$

It is clear that assumptions (H1) of the Theorem 4 is satisfied. On the other hand, for any $t \in [1, e]$, $u, v \in \mathbb{R}$ we have

$$|f(t, u) - f(t, v)| \leq \frac{1}{4}|u - v|,$$

where we have used the trigonometric identity $\arctan(x) - \arctan(y) = \arctan\left(\frac{x-y}{1+xy}\right)$ and $\arctan(x) \leq x, \forall x, y \in \mathbb{R}^+$. Hence condition (H2) holds with $L = \frac{1}{4}$. In addition, by letting $\Phi(t) = \ln t$, we have

$$\mathcal{I}_{a^+}^{\alpha; \psi} \Phi(t) = {}^H \mathcal{I}_{1^+}^{1.5} \ln t = \frac{(\ln t)^{3.5}}{\Gamma(4.5)} \leq \frac{\ln t}{\Gamma(0.5)} := \lambda_{\Phi} \Phi(t).$$

So condition (H6) is satisfied with $\Phi(t) = \ln t$ and $\lambda_{\Phi} = \frac{1}{\Gamma(0.5)} = \frac{1}{\sqrt{\pi}}$. It follows from Theorem 4 that the problem (35) has a unique solution on $[1, e]$, and from Theorem 10, the problem is \mathbb{E}_{α} -Ulam-Hyers-Rassias stable and consequently it is generalized \mathbb{E}_{α} -Ulam-Hyers-Rassias stable.

Example 4. Consider the following problem:

$$(36) \quad \begin{cases} {}^c \mathcal{D}_{0^+}^{\frac{3}{2}} u(t) = e^{-2t}(u^2(t) - 1), & t \in \mathbb{J} := [0, 1], \\ u(0) = 1, & u'(0) = 0. \end{cases}$$

Note that, this problem is a particular case of IVP (1), where

$$\alpha = \frac{3}{2}, \quad \psi(t) = t, \quad f(t, u) = e^{-2t}(u^2 - 1).$$

Obviously, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. On the one hand, taking $v_0(t) = 1$ and $w_0(t) = 1 + t\sqrt{t}$, it is not difficult to verify that v_0, w_0 are lower and upper solutions of (36), respectively, and $v_0 \leq w_0$. So condition (H3) holds. Moreover, for $v_0 \leq u \leq v \leq w_0$ we have

$$(37) \quad f(t, v) - f(t, u) \geq \frac{2}{e^{2t}}(v - u), \quad t \in \mathbb{J}.$$

In view of (37), we can choose $p(t) = \frac{2}{e^{2t}}$ in Theorem 6. Hence, all conditions of Theorem 6 are satisfied and consequently the problem (36) has extremal solutions on $[v_0, w_0]$.

On the other hand, for $v_0 \leq u \leq v \leq w_0$, we have

$$(38) \quad f(t, v) - f(t, u) \leq (t + 2)(v - u), \quad t \in \mathbb{J}.$$

In view of (38), we can choose $\tilde{p}(t) = (t + 2)$ in Theorem 7. Therefore, all conditions of Theorem 7 are satisfied and consequently the problem (36) has unique extremal solutions.

Remark 4. Since the nonlinear term do not satisfy Lipschitz condition, the problem (36) can not be solved by the results in [12, 38].

Example 5. Consider the following initial value problem:

$$(39) \quad \begin{cases} {}^c\mathcal{D}_{0+}^{\frac{3}{2}} u(t) = \frac{\sin t}{2} \left(u(t) + \sqrt{1 + u^2(t)} \right), & t \in [0, +\infty), \\ u(0) = 2, \quad u'(0) = 3. \end{cases}$$

where

$$f(t, u) = \frac{\sin t}{2} \left(u + \sqrt{1 + u^2} \right), \quad \text{for } t \in [0, +\infty), u \in \mathbb{R}.$$

Clearly, $f : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. On the other hand, for any $t \in [0, +\infty), u, v \in \mathbb{R}$ we have

$$\begin{aligned} |f(t, u) - f(t, v)| &= \sin t \left| \frac{1}{2} \left(u - v + \sqrt{1 + u^2} - \sqrt{1 + v^2} \right) \right| \\ &= \sin t \left| \frac{1}{2} (u - v) \left(1 + \frac{u + v}{\sqrt{1 + u^2} + \sqrt{1 + v^2}} \right) \right| \\ &\leq |u - v|. \end{aligned}$$

Hence condition (H2) holds with $L = 1$. Moreover, if we choose, $\beta > 1$, it follows that the mapping \mathcal{T} is a contraction. Hence by Theorem 8 the IVP (39) has a unique solution on $[0, +\infty)$.

REFERENCES

- [1] S. Abbas, M. Benchohra, G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.
- [2] S. Abbas, M. Benchohra, G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.
- [3] S. Abbas, M. Benchohra, J. R. Graef, J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.
- [4] S. Abbas, M. Benchohra, J. E. Lazreg, A. Alsaedi, Y. Zhou, *Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type*, *Advances in Difference Equations*, 180 (2017), 14 pages.
- [5] S. Abbas, M. Benchohra, B. Samet, Y. Zhou, *Coupled implicit Caputo fractional q-difference systems*, *Advances in Difference Equations*, 527 (2019), 19 pages.
- [6] S. Abbas, M. Benchohra, N. Hamidi, J. Henderson, *Caputo-Hadamard fractional differential equations in Banach spaces*, *Fractional Calculus and Applied Analysis*, 21 (4) (2018), 1027–1045.
- [7] S. Abbas, M. Benchohra, S. Hamani, J. Henderson, *Upper and lower solutions method for Caputo-Hadamard fractional differential inclusions*, *Mathematica Moravica*, 23 (1) (2019), 107–118.
- [8] M.S. Abdo, S.K. Panchal, A.M. Saeed, *Fractional boundary value problem with ψ -Caputo fractional derivative*, *Proceedings-Mathematical Sciences*, 129 (5) (2019), 14 pages.

-
- [9] R. P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Applicandae Mathematicae, 109 (3) (2010), 973–1033.
- [10] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Communications in Nonlinear Science and Numerical Simulation, 44 (2017), 460–481.
- [11] R. Almeida, *Fractional Differential Equations with Mixed Boundary Conditions*, Bulletin of the Malaysian Mathematical Sciences Society, 42 (4) (2019), 1687–1697.
- [12] R. Almeida, A.B. Malinowska, M.T.T. Monteiro, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Mathematical Methods in the Applied Sciences, 41 (1) (2018), 336–352.
- [13] R. Almeida, A.B. Malinowska, T. Odziejewicz, *Optimal Leader-Follower Control for the Fractional Opinion Formation Model*, Journal of Optimization Theory and Applications, 182 (3) (2019), 1171–1185.
- [14] R. Almeida, M. Jleli, B. Samet, *A numerical study of fractional relaxation-oscillation equations involving ψ -Caputo fractional derivative*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113 (3) (2019), 1873–1891.
- [15] M. Al-Refai, M. Ali Hajji, *Monotone iterative sequences for nonlinear boundary value problems of fractional order*, Nonlinear Analysis, 74 (11) (2011), 3531–3539.
- [16] A. Ardjouni, H. Boulares, Y. Laskri, *Stability in higher-order nonlinear fractional differential equations*, Acta et Commentationes Universitatis Tartuensis de Mathematica, 22 (1) (2018), 37–47.
- [17] Z. Baitiche, K. Guerbaty, M. Benchohra, Y. Zhou, *Solvability of Fractional Multi-Point BVP with Nonlinear Growth at Resonance*, Natsional cprime naya Akademiya Nauk Armenii. Izvestiya. Matematika, 55 (2) (2020), 126–142.
- [18] M. Benchohra, J. Henderson, D. Seba, *Measure of Noncompactness and Fractional Differential Equations in Banach Spaces*, Communications in Applied Analysis, 12 (4) (2008), 419–428.
- [19] M. Benchohra, J. R. Graef, S. Hamani, *Existence results for boundary value problems with non-linear fractional differential equations*, Applicable Analysis, 87 (7) (2008), 851–863.
- [20] M. Benchohra, S. Hamani, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Analysis 71 (2009), 2391–2396.
- [21] M. Benchohra, S. Bouriah, *Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order*, Moroccan Journal of Pure and Applied Analysis, 1 (1) (2015) 22–37.
- [22] M. Benchohra, S. Bouriah, J.J. Nieto, *Existence of periodic solutions for nonlinear implicit Hadamard's fractional differential equations*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 112 (1) (2018), 25–35.

- [23] M. Benchohra, J.E. Lazreg, *On stability for nonlinear implicit fractional differential equations*, *Le Mathematique*, 70 (2) (2015) 49–61.
- [24] C. Chen, M. Bohner and B. Jia, *Method of upper and lower solutions for nonlinear Caputo fractional difference equations and its applications*, *Fractional Calculus and Applied Analysis*, 22 (5) (2019), 1307–1320.
- [25] N. D. Cong, H. T. Tuan, *Existence, uniqueness, and exponential boundedness of global solutions to delay fractional differential equations*, *Mediterranean Journal of Mathematics*, 14 (5) (2017), 12 pages.
- [26] K. Diethelm and N. J. Ford, *Analysis of fractional differential equations*, *Journal of Mathematical Analysis and Applications*, 265 (2) (2002), 229–248.
- [27] F. Ge, C. Kou, *Stability analysis by Krasnoselskii's fixed point theorem for nonlinear fractional differential equations*, *Applied Mathematics and Computation*, 257 (2015), 308–316.
- [28] R. Gorenflo, A.A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, New York, 2014.
- [29] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, New York, 2003.
- [30] D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract spaces*, Academic Press, Boston, 1982.
- [31] A. Hallaci, H. Boulares and M. Kurulay, *On the study of nonlinear fractional differential equations on unbounded interval*, *General Letters in Mathematics*, 5 (3) (2018), 111–117.
- [32] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
- [33] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of North-Holland Mathematics Studies Elsevier Science B.V. Amsterdam the Netherlands, 2006.
- [34] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, 1978.
- [35] K.D. Kucche, A. Mali, J. Vanterler da C. Sousa, *On the nonlinear Ψ -Hilfer fractional differential equations*, *Computational & Applied Mathematics*, 38 (2) (2019), 25 pages.
- [36] X. Lin and Z. Zhao, *Iterative technique for a third-order differential equation with three-point nonlinear boundary value conditions*, *Electronic Journal of Qualitative Theory of Differential Equations*, 2016 (12) (2016), 1–10.
- [37] J-G. Liu, X.-J. Yang, Y-Y. Feng, P. Cui *New fractional derivative with sigmoid function as kernel and its models*, Preprint December 2019 DOI: 10.13140/RG.2.2.15764.24962.
- [38] M. Matar, *On existence of positive solution for initial value problem of nonlinear fractional differential equations of order $1 < \alpha \leq 2$* , *Acta Mathematica Universitatis Comenianae*, 84 (1) (2015), 51–57.

-
- [39] M. M. Matar, E. S. Abu Skhail, *On stability analysis of semi-linear fractional differential systems*, *Mathematical Methods in the Applied Sciences*, 43 (5) (2020), 2528–2537.
- [40] K.S. Miller, B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
- [41] E.C. Oliveira, J.V.D.C. Sousa, *Ulam–Hyers–Rassias stability for a class of fractional integro-differential equations*, *Results in Mathematics*, 73 (3) (2018), 16 pages.
- [42] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [43] R. Roy and K. H. Kumar *An iterative scheme for a class of fractional order perturbed differential equations*, https://doi.org/10.1007/978-981-13-9939-8_14.
- [44] J. Sabatier, O. P. Agrawal, J. A. T. Machado, *Advances in Fractional Calculus—Theoretical Developments and Applications in Physics and Engineering*, Dordrecht: Springer, 2007.
- [45] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg & Higher Education Press, Beijing, 2010.
- [46] V. Tarasov, *Handbook of Fractional Calculus with Applications*, De Gruyter, Berlin 2019.
- [47] J. Vanterler da Costa Sousa and E. Capelas de Oliveira, *A Gronwall inequality and the Cauchy-type problem by means of ψ -Hilfer operator*, *Differential Equations & Applications* 11 (1) (2019), 87–106.
- [48] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *Existence, uniqueness, estimation and continuous dependence of the solutions of a nonlinear integral and an integrodifferential equations of fractional order*, arXiv:1806.01441, (2018).
- [49] G. Wang, W. Sudsutad, L. Zhang, J. Tariboon, *Monotone iterative technique for a nonlinear fractional q -difference equation of Caputo type*, *Advances in Difference Equations*, 211 (2016), 11 pages.
- [50] J. Wang, L. Lv, Y. Zhou, *Ulam stability and data dependence for fractional differential equations with Caputo derivative* *Electronic Journal of Qualitative Theory of Differential Equations*, 2011, (63) (2011), 1–10.
- [51] J. Wang, Y. Zhou, *Mittag-Leffler-Ulam stabilities of fractional evolution equations*. *Applied Mathematics Letters*, 25 (4) (2012), 723–728.
- [52] J. Wang, X. Li, *E_α -Ulam type stability of fractional order ordinary differential equations*, *Journal of Applied Mathematics and Computing*, 45 (2014), 449–459.
- [53] J. Wang, Y. Zhang, *Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations*, *Optimization* 63 (8) (2014), 1181–1190.
- [54] W. Yang, *Monotone iterative technique for a coupled system of nonlinear Hadamard fractional differential equations*, *Journal of Applied Mathematics and Computing*, 59 (2019), 585–596.

- [55] S. Zhang, *Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives*, *Nonlinear Analysis* 71 (2009), 2087–2093.
- [56] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore 2014.
- [57] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Elsevier/Academic Press, London 2016.

CHOUKRI DERBAZI

LABORATORY OF MATHEMATICS AND APPLIED SCIENCES
UNIVERSITY OF GHARDAIA
PO BOX 455, 47000 GHARDAIA
ALGERIA
E-mail address: choukriedp@yahoo.com

ZIDANE BAITICHE

LABORATORY OF MATHEMATICS AND APPLIED SCIENCES
UNIVERSITY OF GHARDAIA
PO BOX 455, 47000 GHARDAIA
ALGERIA
E-mail address: baitichezidane19@gmail.com

MOUFFAK BENCHOHRA

LABORATORY OF MATHEMATICS
DJILLALI LIABES UNIVERSITY OF SIDI BEL-ABBÈS
PO BOX 89, 22000, SIDI BEL-ABBÈS
ALGERIA
E-mail address: benchohra@yahoo.com

GASTON N'GUÉRÉKATA

DEPARTMENT OF MATHEMATICS
MORGAN STATE UNIVERSITY
1700 E. COLD SPRING LANE
BALTIMORE M.D. 21252
USA
E-mail address: Gaston.N'Guerekata@morgan.edu